Discrete Optimization

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Dealing with a (NP-)hard problem

Bounding the solution

- Need of a primal bound (upper bound in the case of a minimization problem)
- Need of a dual bound (lower bound in the case of a minimization problem)

When the two bounds meet, we have a proof of optimality.

The primal bound

A primal bound is a lower bound to the value of the optimal solution for a maximization.

A primal bound is an upper bound to the value ofthe optimal solution for a minimization.

How to find a primal bound ?

By finding a feasible solution to the problem.
The best possible primal bound is given by the optimal solution.

The dual bound

A dual bound is an upper bound to the value of the optimal solution for a maximization.A dual bound is a lower bound to the value of the optimal solution for a minimization.

How to find a dual bound ?

There are several ways.
We will cover two ways : through relaxations, through Lagrangian duality.

Relaxations

Consider an optimization problem :

$$\min c(x)$$
$$\text{s.t. } x \in X.$$

A relaxation is an (easier) optimization problem for which the value of the optimal solution is guaranteed to be lower than that of the initial problem.

Ways to obtain a relaxation

- Enlarge the feasible set $Y \supseteq X$ and solve

$$\min c(x)$$
$$\text{s.t. } x \in Y.$$

- Replace the objective function $c(x)$ by a lower value $d(x)$ for every feasible x, i.e. $d(x) \leq c(x)$ for all $x \in X$.

$$\min d(x)$$
$$\text{s.t. } x \in X.$$

- Or combine the two.

Note : if we replace the feasible set by a smaller set $Y \subseteq X$, we talk about a restriction which may be useful to find primal bounds.

The linear programming relaxation

For a mixed-integer optimization problem of the form

$$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t. } & Ax + Gy \leq b \\ & x \in \mathbb{Z}^n_+, y \in \mathbb{R}^m_+, \end{aligned}$$

the linear programming relaxation consists in replacing the integrality constraints by simple nonnegativity constraints :

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The set of feasible solutions is now larger than before.

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When the linear programming relaxation tells you everything

Two lucky cases allow us to solve a problem just with the linear programming relaxation.

Proposition

- If the linear programming relaxation of a mixed-integer optimization problem is infeasible, then the mixed-integer optimization problem is infeasible as well.

- If an optimal solution of the linear programming relaxation of a mixed-integer optimization problem is integral then it is also optimal for the mixed-integer optimization problem.

Branch-and-Bound

Idea : enumerate but using the information of the linear relaxation.

LP Solution : $x^{1*} = (\frac{265}{79}, \frac{160}{79})$ with optimal cost 12.79

Branch-and-Bound

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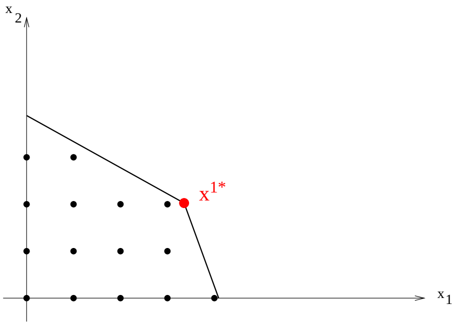

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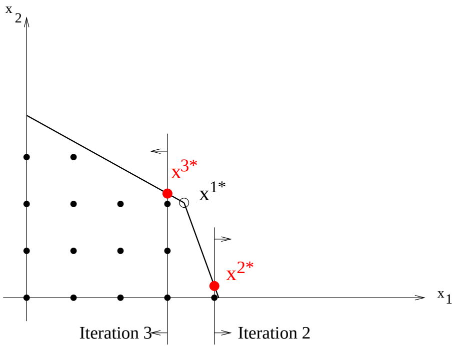

LP Solution : $x^{1*} = (\frac{265}{79}, \frac{160}{79})$ with optimal cost 12.79
2 branches are created : either $x_1 \geq 4$ or $x_1 \leq 3$
Branch 1 : $x_1 \geq 4$: $x^{2*} = (4, \frac{1}{4})$ with optimal cost 8.75
Prune by bound if we suppose $x = (0, 3)$ with cost 9 is known.

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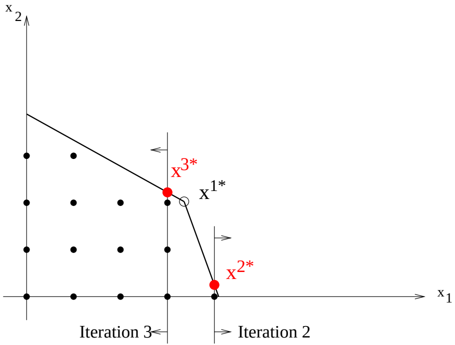

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Branch 2 : $x_1 \leq 3$: $x^{3*} = (3, \frac{20}{9})$ with optimal cost 12.67

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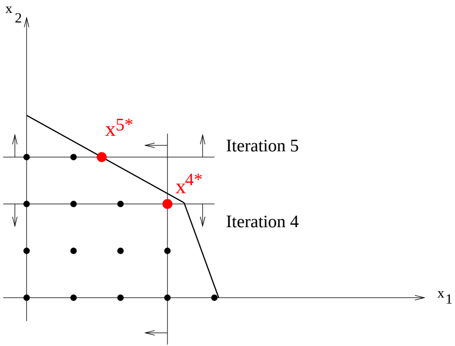


LP Solution : $x^{1*} = (\frac{265}{79}, \frac{160}{79})$ with optimal cost 12.79

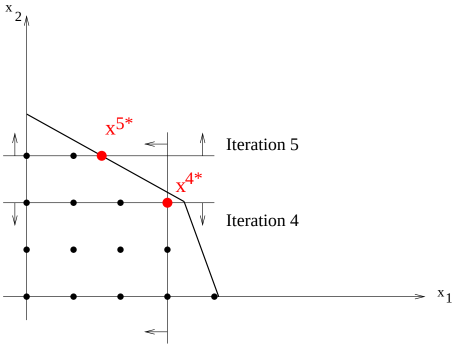
2 further branches are created : either $x_2 \leq 2$ or $x_2 \geq 3$

Branch 2.1 : $x_2 \leq 2$: $x^{4*} = (3, 2)$ with optimal cost 12

Prune by optimality

Branch-and-Bound

Idea : enumerate but using the information of the linear relaxation.



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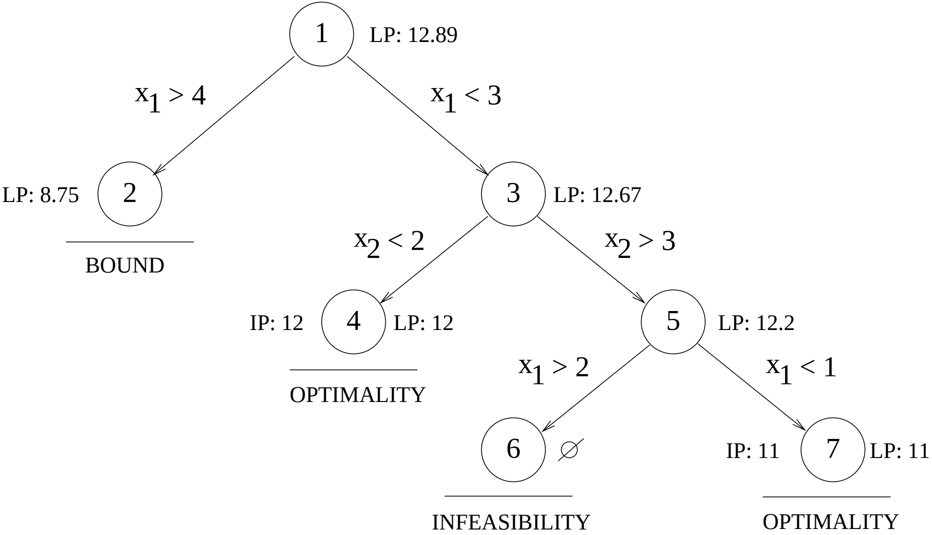
Prune by optimality

Branch 2.2 : $x_2 \geq 3$: $x^{5*} = (\frac{8}{5}, 3)$ with optimal cost 12.2
2 further branches : $x_1 \leq 1$ which gives $(1, 3)$ (prune by optimality and bound) and $x_1 \geq 2$ (prune by infeasiblity)

Summary of the branch-and-bound tree



Branch-and-Bound

Remarks

- Opportunities to prune the search :
  By bound, By optimality, By infeasibility
- Need of a good primal bound in the beginning
- Different strategies for the node selection :
  depth-first-search (good to find quickly primal solutions)
  breadth-first-search (good to increase the dual bound)
- Different strategies for variable selection :
  Most fractional variable or least fractional variable
  Take advantage of the history of branching
  Look ahead for best improvement in the bound : strong branching