Discrete Optimisation
Exercise Session 8: Cuts

13th November 2015

Exercise 1 (lifting covers). For the set $X = \{(u, v, w, x, y, z) \in \mathcal{B}^6 \mid 12\,u + 9\,v + 7\,w + 5\,x + 5\,y + 3\,z \leq 14\}$ and the cover $w + y + z \leq 2$:

1. Determine whether the cover is a facet of $X \cap \{(u, v, w, x, y, z) \in \mathcal{B}^6 \mid u = v = x = 0\}$.
2. Lift the inequality for X.

Exercise 2 (optimal subtour elimination generation). The generalised subtour elimination constraint can be used to formulate the prize-collecting travelling salesman problem: with respect to the travelling salesman problem, travelling through an edge e has a cost c_e, and visiting a city j allows them to make a profit of f_j; not all cities have to be visited, but the salesman still must follow a cycle, starting at the first city.

1. Give a MILP model for the prize-collecting travelling salesman problem.
2. What could be practical problems of implementing this formulation?
3. Solve the following instance of the prize-collecting travelling salesman problem with lazy constraint generation.

$$c_e = \begin{pmatrix} 0 & 4 & 3 & 3 & 5 & 2 & 5 \\ 4 & 0 & 5 & 3 & 3 & 4 & 7 \\ 3 & 5 & 0 & 4 & 6 & 0 & 4 \\ 3 & 3 & 4 & 0 & 4 & 4 & 6 \\ 5 & 3 & 6 & 4 & 0 & 5 & 8 \\ 2 & 4 & 0 & 4 & 5 & 0 & 3 \\ 5 & 7 & 4 & 6 & 8 & 3 & 0 \end{pmatrix},$$

$$f_j = \begin{pmatrix} 2 & 4 & 1 & 3 & 7 & 1 & 7 \end{pmatrix}.$$

Exercise 3 (solving a knapsack with covers and lifting). Solve the following knapsack instance (knapsack capacity: twenty kilograms) using a computer and only a LP solver.

Item	1	2	3	4	5	6	7
Profit (€)	4	5	6	6	6	10	15
Weight (kg)	5	5	6	7	8	12	16

1. Write the corresponding integer linear program. Find a feasible solution: this is a lower bound on the objective function. You should get 17.
2. Write its linear relaxation and solve it to optimality. The objective value is a first upper bound; you should get 19.4375.
3. Find as many violated cover inequalities as possible by the following process:

    (a) Solve the linear relaxation,

(b) Find (at least) one cover inequality that is violated by the solution to this LP

After adding four constraints, you could get the upper bound 18.6363 and the optimal solution

$$(0, 0.9090, 0.9090, 0.9090, 0, 0.1818, 0.0909)\,.$$

4. Extend some cover inequalities by inspection (try to add a new variable into the mix without changing the right-hand side).
   For example, with the inequality $x_2 + x_3 + x_6 \leq 2$, you can generate $x_2 + x_3 + x_6 + x_7 \leq 2$, as only two objects (at most) amongst them can be taken at once: if you take x_2 and x_3, then it is no more possible to take x_6 or x_7.
   After adding three constraints (the previous one and two slight variations thereof), you could get the upper bound 18.6 and the optimal solution
   $$(0.4, 1, 1, 1, 0, 0, 0)\,.$$

5. Show that an extended inequality by the previous procedure is less strong than one generated by lifting. To do this, start from $x_1 + x_2 + x_3 + x_4 \leq 3$, generate one valid inequality by extension, and another one by lifting, compare them by optimising the corresponding relaxations (is one bound tighter than the other?).
   After adding the lifted cover, you could get the upper bound 17.6666 and the optimal solution
   $$(0, 0.3333, 1, 1, 0.6666, 0, 0)\,.$$

6. Add new cover inequalities and lift them until the problem is solved; you should get an optimal value of 17 with the optimal solution
   $$(0, 1, 1, 1, 0, 0, 0)\,.$$

Remark. Multiple paths are possible, getting different intermediate results. Some may be shorter than others; the intermediate results are shown for the shortest one.