Discrete Optimisation
Exercise Session 5: Formulation Comparison

18th October 2016

After this session, you should be able to solve all exercises of Chapter 4 in the exercises book.

Exercise 1 (discrete facility location — 4.1.1)**.** The uncapacitated facility location problem deals with the optimal opening of facilities (their position cannot be changed) to meet some demand (always one unit) while minimising the total cost (opening facilities, producing the goods, delivering them to the customers). There is no bound on the amount of goods a facility can produce. A client's demand is always met by only one facility.

1. Propose two formulations for the uncapacitated facility location.
   Hint: one has an aggregated constraint, not the other one.
2. Is a formulation stronger than the other, or are they equal?
3. Implement both formulations and compare the solving times when increasing the size of the problem. Is one better than the other? Does it correspond to your expectations?

Exercise 2 (exams schedule — 4.1.2)**.** We want to create a schedule for the exam session. In order to do so, we formulate the problem of finding a time slot for each exam in such a way that no student has more than one exam on the same day. We define binary variables as

$$\begin{aligned} x_{it} &= 1 \text{ if exam } i \text{ takes place on day } t \\ &= 0 \text{ otherwise.} \end{aligned}$$

We denote by L the set of courses, by T the set of days and by S the set of all students. For a given student $k \in S$, we denote by $C\left(k\right)$ the index set of all courses followed by student k. To encode the constraints, we suggest two options:

1. $$\begin{aligned} \sum_{t \in T} x_{it} &= 1 \qquad \text{for all } i \in L, t \in T \\ \sum_{i \in C(k)} x_{it} &\leq 1 \qquad \text{for all } k \in S, t \in T. \end{aligned}$$

2. $$\begin{aligned} \sum_{t \in T} x_{it} &= 1 \qquad \text{for all } i \in L, t \in T \\ x_{it} + x_{jt} &\leq 1 \qquad \text{for all } \left(i, j\right) \in C(k) \times C(k)\,, k \in S, t \in T, i \neq j. \end{aligned}$$

Determine between (a) and (b) which one is the best formulation and prove that it is the case.

Exercise 3 (travelling salesman problem — 4.2.1)**.** Given a list of cities and the distances between each pair, the undirected travelling salesman problem finds the shortest path that visits each city exactly once while returning to the original city. It has applications outside operational research; for example, to manufacture microchips, in order to minimise delays, every component must be placed as closely as possible to the other ones; the components' position can also be optimised.

1. Propose two formulations for the undirected travelling salesman.
   Hint: one eliminates subtours and is called subtour elimination, the other imposes connexity and is named cut-set.

2. Prove that the two formulations are equivalent or that one is stronger than the other. To this end, follow these steps.

   (a) For a set of vertices $S \subsetneq V$, what kind of relationship is there between the set of outgoing edges $\delta(S)$, the set of edges within S $E(S)$, and the whole set of edges $E = E(V)$?

   (b) What can you say about $P_{\text{cut}} \subseteq P_{\text{sub}}$?

   (c) What can you say about $P_{\text{sub}} \subseteq P_{\text{cut}}$?

3. Implement both formulations and compare the solving times when increasing the size of the problem. Is one better than the other? Does it correspond to your expectations?